NONLINEAR PROBLEM OF THE THEORY OF ELASTICITY ON THE SPLITTING OF A PLATE

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This article examines a two-dimensional problem of the splitting of a nonlinearly elastic body made of a material of the harmonic type [1] with the assumption that the acting forces retain their magnitude and direction during deformation [2].

Let a rigid semi-infinite wedge of constant thickness 2h be driven into an infinite plate made of the above-described brittle material. The material has the elastic characteristics E (elastic modulus) and ν (Poisson's ratio). Driving of the wedge into the plate forms a rectilinear crack ahead of the wedge, the length of the crack being designated here as L. We then assume that cohesive forces of intensity G act in the tip region of the crack. We introduce the universal constant K of the material, called the cohesion modulus and determined by the formula [3]

$$K = \int_0^d \frac{G(t)\,dt}{\sqrt{t}}, \tag{1}$$

where d is the width of the tip region.

Subsequent discussions will be based on the following assumptions, which were formulated in [3, 4]: 1) The size of the crack-tip zone is negligibly small compared to the size of the entire crack; 2) the distribution of the displacement in the tip zone is independent of the acting loads; 3) the stresses on the crack edges are finite.

Along with these assumptions, we will suppose that there is no friction in the contact region between the wedge and plate.

This problem was solved in the linear case in [3]. Below we examine the problem in a nonlinear formulation in the sense that the material surrounding the crack is taken to be nonlinearly elastic as described above.

For the physical region we take the plane of the variable z = x + iy, cut out along the positive part of the Oy axis. We assume that the wedge acts on the segments [L; ∞] of this axis. The length of the crack formed in front of the wedge is assumed to be so much greater than h that the boundary conditions for the entire crack surface can be referred to the above cut-out section. Thus, we can use this method of modeling for the present case as well, since the nonlinear character of the problem is determined by the behavior of the harmonic elastic material surrounding the wedge (see [1, 2] and the numerical data at the end of this article). After considering this, we can represent the boundary conditions of the problem in the form [5]

$$X_y = 0 \text{ on } [0;\infty[,\ X_x = G(y) \text{ on } [0;d],\ X_x = 0 \text{ on }]d;L]; \tag{2}$$

$$X_x = -f(y),\ u(x+0,y) = h,\ u(x-0,y) = -h \text{ on }]L;\infty[, \tag{3}$$

where X_x, Y_y, and X_y are components of the stress tensor; u and v are components of the displacement vector; f(y) is a real function specified on [L; ∞] and characterizing the forces acting on the wedge. This function is unknown beforehand and is liable to determination.

We will solve the problem using complex representations of the fields of the elastic elements in the region outside the crack through two functions φ(z) and ψ(z) which are analytic in the investigated region (see [6]):

$$X_x + Y_y + 4\mu = \frac{\lambda+2\mu}{\sqrt{I}} q\Omega(q), \quad Y_y - X_x - 2iX_y = -\frac{4(\lambda+2\mu)}{\sqrt{I}} \frac{\Omega(q)}{q} \frac{\partial z^*}{\partial z} \frac{\partial z^*}{\partial \bar{z}}; \tag{4}$$

$$u + iv = \frac{\mu}{\lambda+2\mu} \int \varphi'^2(z)\,dz + \frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)}{\overline{\varphi'(z)}} + \overline{\psi(z)}\right] - z; \tag{5}$$

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$$\frac{\partial z^*}{\partial z}=\frac{\mu}{\lambda+2\mu}\varphi'^2(z)+\frac{\lambda+\mu}{\lambda+2\mu}\frac{\varphi'(z)}{\overline{\varphi'(z)}},\quad \frac{\partial z^*}{\partial \bar z}=-\frac{\lambda+\mu}{\lambda+2\mu}\left[\frac{\varphi(z)\overline{\varphi''(z)}}{\overline{\varphi'^2(z)}}-\overline{\psi'(z)}\right], \tag{6}$$

where

$$\sqrt{I}=\frac{\partial z^*}{\partial z}\frac{\partial \bar z^*}{\partial \bar z}-\frac{\partial z^*}{\partial \bar z}\frac{\partial \bar z^*}{\partial z};\quad q=2\left|\frac{\partial z^*}{\partial z}\right|;\quad \Omega(q)=q-\frac{2(\lambda+\mu)}{\lambda+2\mu}; \tag{7}$$

λ and μ are the Lamé constants.

In the case being examined, the function $\psi(z)$ is bounded at infinity, while $\varphi(z)$ has the following asymptote at large [z] [6]

$$\varphi(z)=z+\varphi_0(z), \tag{8}$$

where $\varphi_0(z)$ is a function which is analytic and bounded at $z=\infty$.

It is easy to see that adherence to the first condition of (2) leads on the basis of (4) and (6) to the relation

$$\overline{\varphi(y)}\varphi''(y)/\varphi'^2(y)-\psi'(y)=0\quad \text{at}\quad y\geqslant L. \tag{9}$$

We will differentiate Eq. (5) with respect to y and take (9) into account in the resulting relation. Then after some elementary transformations we obtain the following on [L; ∞]

$$v_y'-iu_y'=\varphi'^2(y)\left[\frac{\mu}{\lambda+2\mu}+\frac{\lambda+\mu}{\lambda+2\mu}\frac{1}{|\varphi'^2(y)|}\right]-1,\quad y\in[L;\infty[. \tag{10}$$

Then from (4) we arrive at the following relation by means of (3), (6), and (9)

$$X_x=2\mu(\lambda+\mu)[|\varphi'^2(y)|-1]/[\mu|\varphi'^2(y)|+\lambda+\mu]\quad \text{on}\quad [L;\infty[. \tag{11}$$

After this, we use the relation

$$z=x+iy=\omega(\zeta)=i\zeta^2 \tag{12}$$

to map the investigated physical region conformally and mutually unambiguously onto the lower half-plane S of the plane of the variable $\zeta=\xi+i\eta$ $(\zeta=\rho e^{i\theta})$ and we keep the former notation for the functions being examined.

We conclude on the basis of (8) and (12) that the behavior of $\varphi'(\zeta)$ with sufficiently large $|\zeta|$ is characterized by the formula

$$\varphi'(\zeta)=2i\zeta+O(\zeta^{-2}). \tag{13}$$

Now let us return to Eq. (11), from which we find the following by means of (3) and (12)

$$\left|\frac{\varphi'(\sigma)}{\omega'(\sigma)}\right|=\sqrt{(\lambda+\mu)[2\mu-f(\sigma^2)]/\mu[2(\lambda+\mu)+f(\sigma^2)]} \tag{14}$$

$$\text{on}\quad \gamma=]-\infty;-\sqrt{L}[\cup]\sqrt{L};\infty[.$$

Considering the known properties of functions holomorphic in the region and taking account of (13), from (14) we obtain

$$\varphi'(\zeta)=2i\zeta\exp\frac{1}{\pi i\zeta}\int\limits_{\gamma}\frac{\sigma F(\sigma^2)\,d\sigma}{\sigma-\zeta}\quad \text{with}\quad \zeta\in S, \tag{15}$$

where

$$F(\sigma^2)=\frac{1}{2}\ln\frac{\lambda+\mu}{\mu}\frac{2\mu-f(\sigma^2)}{2(\lambda+\mu)+f(\sigma^2)}. \tag{16}$$

Now let us return to (4) and require satisfaction of the condition of finiteness of the stresses at the ends of the crack. In the transformed region this condition is equivalent to the equality $\varphi'(0)=0$, which on the basis of (1) and (15) leads to the relation

$$\frac{1}{\pi i}\int\limits_{-\infty}^{-\sqrt{L}}\frac{\sigma F(\sigma^2)\,d\sigma}{\sigma-\zeta}+\frac{1}{\pi i}\int\limits_{\sqrt{L}}^{\infty}\frac{\sigma F(\sigma^2)\,d\sigma}{\sigma-\zeta}-\frac{1}{\pi i}\int\limits_{-\sqrt{d}}^{\sqrt{d}}\frac{\sigma F(\sigma^2)\,d\sigma}{\sigma-\zeta}=0\quad \text{at}\quad \zeta=0,$$

or

$$\frac{2}{\pi i}\int_{\sqrt{L}}^{\infty} F(\sigma^2)\,d\sigma - \frac{2}{\pi i}\int_{0}^{\sqrt{d}} G(\sigma^2)\,d\sigma = 0.$$

From here, with allowance for (1), we obtain the following (in the old coordinates) on the basis of hypotheses 1 and 2:

$$\int_{L}^{\infty}\frac{F(y)\,dy}{\sqrt{y}} = K. \tag{17}$$

Equation (17) expresses the condition of equilibrium of the crack, but its left side is unknown because f and, thus, F have not yet been determined.

For this, we return to Eq. (10) and insert into it the values obtained from (12) and (15). Then after certain manipulations and application of the familiar Sokhotskii–Plemelj relations, on the basis of (3) we obtain

$$\frac{\lambda+\mu}{\mu}\,\frac{2\mu - f(\sigma_0^2)}{2(\lambda+\mu)+f(\sigma_0^2)}\,\mathrm{Im}\exp\frac{1}{\pi i}\int_{\gamma}\frac{\sigma F(\sigma^2)\,d\sigma}{\sigma-\sigma_0} = 0,$$

or

$$\int_{\gamma}\frac{\sigma F(\sigma^2)\,d\sigma}{\sigma-\sigma_0} = 0. \tag{18}$$

This equality represents a singular integral equation for determining the function F and γ and, as condition (17), it is outwardly similar to the corresponding relations of the linear classical theory. The difference is that the function F(y) in the case being examined is determined through f(y) by nonlinear relation (16) (in the linear theory, F(y) = f(y)).

With allowance for (12), singular integral equation (18) has the solution

$$F(y) = A\sqrt{L}/\sqrt{y(y-L)}, \tag{19}$$

where A is a yet-unknown constant. To determine it, we introduce (19) into the left side of (17). Then after some elementary calculations we obtain $A = K/\pi$ and, thus, the solution of (19) will have the form

$$F(y) = K\sqrt{L}/[\pi\sqrt{y(y-L)}]. \tag{20}$$

With allowance for (20), we find the values of f(y) from (16) in the form

$$f(y) = 2\mu\left[\exp\frac{K\sqrt{L}}{\pi\sqrt{y(y-L)}} - 1\right]\Bigg/\left[1 + \frac{\mu}{\lambda+\mu}\exp\frac{K\sqrt{L}}{\pi\sqrt{y(y-L)}}\right].$$

It must be remembered that in accordance with the linear theory

$$f(y) = K\sqrt{L}/[\pi\sqrt{y(y-L)}].$$

To determine the length L of the crack being examined, we insert (15) into (10) and require that as $y \to \infty$ the left side of this relation is equal in absolute value to h. Then, as can easily be shown, for large y we should have

$$F(y) = E\arcsin h/[2\pi(1-\nu^2)y] + O(y^{-2}). \tag{21}$$

By comparing (20) and (21) we find the sought formula in the form

$$L = E^2(\arcsin h)^2/(4(1-\nu^2)^2K^2). \tag{22}$$

In accordance with the classical linear theory, as is known,

$$L = E^2h^2/(4(1-\nu^2)^2K^2. \tag{23}$$

Equations (15), (20), and (22) also determine the sought function (15), after which the rest of the unknowns of the problem are found [6].

It follows from comparison of (22) and (23) that

$$\delta = L_n/L_l = (\arcsin h/h)^2.$$

Table 1 presents values for this relation with different h. It is evident from the table that the length of the crack formed ahead of the wedge is greater according to the nonlinear theory than according to the linear theory.

TABLE 1

h	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
δ	1,0040	1,0080	1,0140	1,0217	1,0316	1,0432	1,0582	1,0760	1,0966

TABLE 2

h	0,1	0,15	0,2	0,25	0,3	0,4	0,5
L	18,40 18,40	40,41 41,77	73,61 74,72	115,02 117,59	165,63 170,78	294,45 311,57	470,08 504,62

TABLE 3

h	0,1	0,15	0,2	0,25	0,3	0,4	0,5
L	20,78 20,78	46,77 47,17	83,12 84,37	129,87 132,58	187,02 192,84	332,48 351,81	519,50 569,79

As an example, we also calculated the lengths of a rectilinear crack for different materials with different values of h. Tables 2 and 3 show results of the calculations for steels 4330 with an elastic modulus $E = 2 \cdot 10^6$ kg/cm^2 and a cohesive modulus $K = 2.5 \cdot 10^4$ kg/cm$^{3/2}$ (Poisson's ratio $\nu = 0.26$). Also shown are results for aluminum 2219-T87 with the characteristics $E = 0.8 \cdot 10^6$ kg/cm^2 and $K = 10^4$ kg/cm$^{3/2}$ ($\nu = 0.35$) [4]. Here, Tables 2 and 3 show the values obtained from the linear theory first and the values from the nonlinear theory second.

The data shows that the nonlinear theory leads to an increase in crack length compared to the linear (classical) theory. The increase is greater, the greater the thickness of the rigid wedge. In the first case, the difference is 2.5, 3.1, and 7.2% with h = 0.15, 0.3, and 0.5. In the second case, the difference is 0.9, 3.2, and 9.8% with h = 0.15, 0.3, and 0.5–0.9.

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